Mechanics of Solids - Review

Dynamics of a continuous media

- In dynamics we study the cause of the motion of a continuum.
- We introduce the laws of motion and conservation of Mass Momentum and Angular Momentum.
- These conservation principles lead to the equations of motion and the definition of the stress tensors.

From the book: Mechanics of Continuous Media: an Introduction
J Botsis and M Deville, PPUR 2018.
Solutions: https://www.epflpress.org/produit/908/9782889152810/mechanics-of-continuous-media

Continuum mechanics review: Dynamics

CONSERVATION LAWS

We consider a quantity $\varphi(\boldsymbol{x},t)$ occupying a volume $\omega(t)$ or ω of a body in motion with velocity $\boldsymbol{v}(\boldsymbol{x},t)$.

To establish the conservation of a given quantity in spatial description:

$$I(t) = \int_\omega \varphi(\boldsymbol{x},t)\,dx_1\,dx_2\,dx_3$$

the variation of the time derivative of that quantity is investigated:

$$\frac{DI(t)}{Dt} = \frac{d}{dt}\int_\omega \varphi(\boldsymbol{x},t)\,dx_1\,dx_2\,dx_3$$

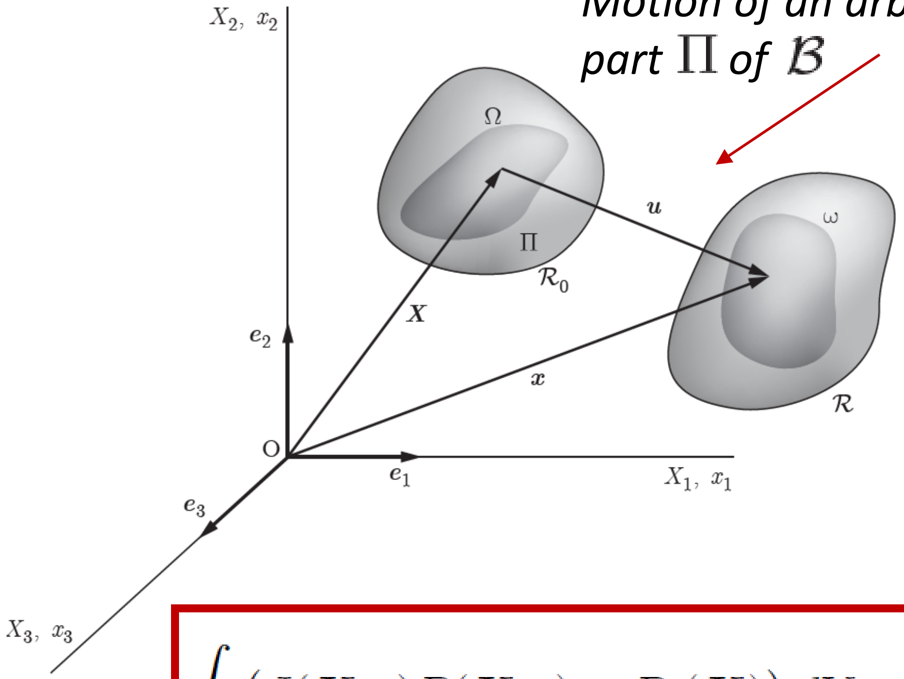
The analysis of this derivative results in the following integral known as **Raynolds Transport Theorem:**

$$\frac{DI(t)}{Dt} = \int_\omega \left(\frac{D\varphi(\boldsymbol{x},t)}{Dt} + \varphi(\boldsymbol{x},t)\nabla\cdot\boldsymbol{v}(\boldsymbol{x},t)\right)dx_1\,dx_2\,dx_3$$

It plays an essential role (together with the divergence theorem) in the models of continuous media.

Continuum mechanics review: Dynamics

CONSERVATION OF MASS IN MATERIAL DESCRIPTION



Motion of an arbitrary part Π *of* $\mathcal{B}$

initial mass

initial mass density

$$m(\Omega) = \int_\Omega P_0(\boldsymbol{X})\, dV = \int_\Omega P_0(\boldsymbol{X})\, dX_1\, dX_2\, dX_3$$

current mass

current mass density

$$m_t(\omega) = \int_\omega \rho(\boldsymbol{x}, t)\, dv = \int_\omega \rho(\boldsymbol{x}, t)\, dx_1\, dx_2\, dx_3\ ,$$

Principle of conservation of mass : $m_t(\omega) = m(\Omega)$

$$\int_\omega \rho(\boldsymbol{x}, t)\, dv = \int_\Omega P_0(\boldsymbol{X})\, dV$$

We change variables on the LHS integral and considering:

$$P(\boldsymbol{X}, t) = \rho\big(\chi(\boldsymbol{X}, t), t\big)$$

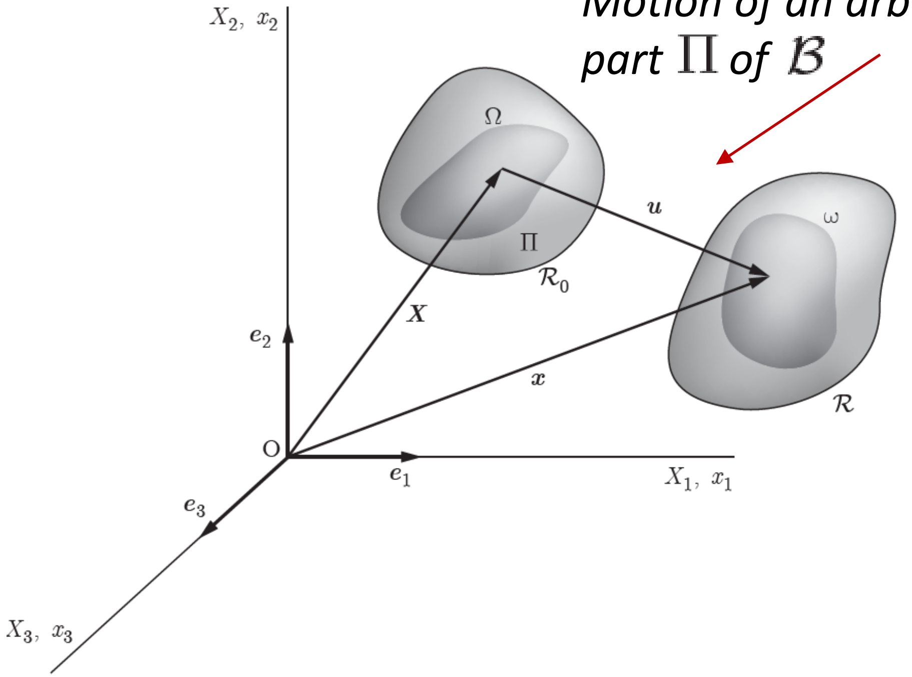
$$\int_\Omega \big(J(\boldsymbol{X}, t)P(\boldsymbol{X}, t) - P_0(\boldsymbol{X})\big)\, dV = 0$$

or: $J(\boldsymbol{X}, t)P(\boldsymbol{X}, t) = P_0(\boldsymbol{X})$

the medium is defined as **incompressible** when : $J(\boldsymbol{X}, t) = 1$

Continuum mechanics review: Dynamics

CONSERVATION OF MASS IN SPATIAL DESCRIPTION



Motion of an arbitrary part Π *of* $\mathcal{B}$

The time derivative of: $\int_\omega \rho(\boldsymbol{x},t)\,dv = \int_\Omega P_0(\boldsymbol{X})\,dV$

and use of the Reynolds theorem results in :

$$\frac{d}{dt}\int_\omega \rho(\boldsymbol{x},t)\,dv = 0$$

or $$\int_\omega \left(\frac{D\rho(\boldsymbol{x},t)}{Dt} + \rho(\boldsymbol{x},t)\,\mathrm{div}\,\boldsymbol{v}(\boldsymbol{x},t)\right)dv = 0$$

defined as the global form of the mass conservation in spatial description.

In local form the **principle of conservation of mass is** :

$$\frac{D\rho(\boldsymbol{x},t)}{Dt} + \rho(\boldsymbol{x},t)\,\mathrm{div}\,\boldsymbol{v}(\boldsymbol{x},t) = 0$$

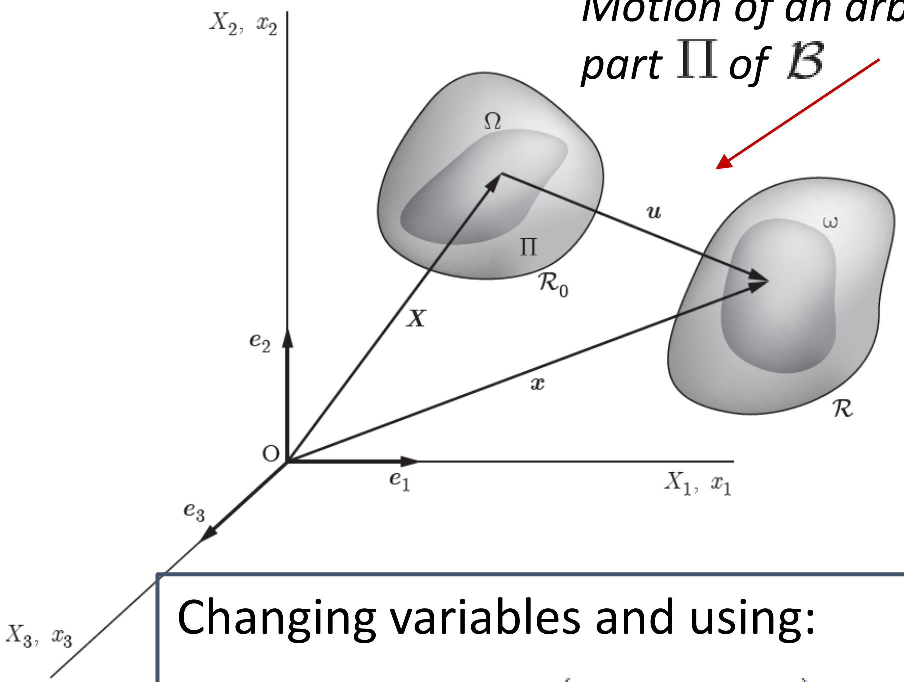
For an **incompressible material** : $\mathrm{div}\,\boldsymbol{v} = \dfrac{\partial v_i}{\partial x_i} = 0$

and $D\rho(\boldsymbol{x},t)/Dt = 0$

The material form of the conservation is used in solid mechanics while the spatial decription in fluid mechanics.

Continuum mechanics review: Dynamics

BODY FORCES



Motion of an arbitrary part Π *of* $\mathcal{B}$

The time volume force acting on Π at time *t* :

$$\boldsymbol{f}^b(\omega, t) = \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)\, dv$$

Here $\boldsymbol{b}(\boldsymbol{x}, t)$ is a vector function defined in $\mathcal{R}$ and called **spatial volume force density.**

Thre material form of this force is:

$$\boldsymbol{F}^b(\Omega, t) = \int_\Omega P_0(\boldsymbol{X})\boldsymbol{B}(\boldsymbol{X}, t)\, dV.$$

In both decritions they express the same quantity thus, we have:

$$\boldsymbol{f}^b(\omega, t) = \boldsymbol{F}^b(\Omega, t)$$

or

$$\int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)\, dv - \int_\Omega P_0(\boldsymbol{X})\boldsymbol{B}(\boldsymbol{X}, t)\, dV = 0$$

Changing variables and using:

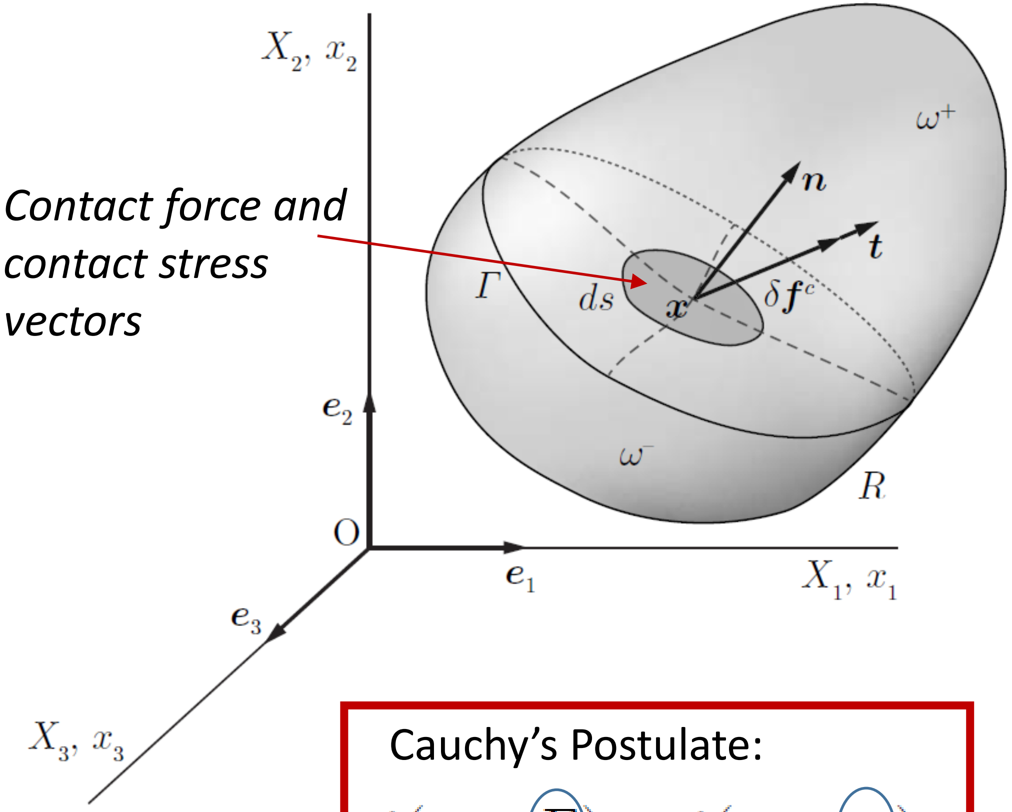
$$P(\boldsymbol{X}, t) = \rho(\chi(\boldsymbol{X}, t), t))$$

We obtain for the **volume force density:**

$$\boldsymbol{B}(\boldsymbol{X}, t) = \boldsymbol{b}(\chi(\boldsymbol{X}, t), t)$$

Continuum mechanics review: Dynamics

CONTACT (SURFACE) FORCES



Contact force and contact stress vectors

Contact forces can describe

1: the interaction between two interior parts of a body $\mathcal{B}$ separated by a surface (i.e., internal cohesive forces).

2: the action of external bodies directly in contact with the boundary of $\mathcal{B}$

We consider the body in two parts $\prod^-$; $\prod^+$ with volumes:

$\omega^- \subset \mathcal{R}$ and $\omega^+ \subset \mathcal{R}$

The two parts are separated by a boundary Γ .

At a time t, the action of $\prod^+$ on $\prod^-$, accros $\delta s(\boldsymbol{x})$ is $\delta \boldsymbol{f}^c(\boldsymbol{x}, t, \Gamma)$

$$\Longrightarrow \quad \boldsymbol{t}(\boldsymbol{x}, t, \Gamma) = \lim_{\delta s \to 0} \frac{\delta \boldsymbol{f}^c(\boldsymbol{x}, t, \Gamma)}{\delta s(\boldsymbol{x})}$$

The limit, when it exists, is called **spatial stress vector**, or **surface stress vector**:

Cauchy's Postulate:

$$\boldsymbol{t}(\boldsymbol{x}, t, \Gamma) = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$$

The action of $\prod^+$ on $\prod^-$ is:

$$\boldsymbol{f}^c(\Gamma, t) = \int_\Gamma \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds$$

Continuum mechanics review: Dynamics

CONSERVATION OF MOMENTUM

In physics, the momentum of a particle with mass *m* is defined by:

$$\overline{\boldsymbol{m}} = m\boldsymbol{v} \qquad \overline{m}_i = m v_i$$

For a part Π of a body $\mathcal{B}$ initially at $\mathcal{R}$ and currently at $\mathcal{R}_t$, the momentum is:

$$\overline{\boldsymbol{m}}(\omega, t) = \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\, dv$$

$$\overline{m}_i(\omega, t) = \int_\omega \rho(\boldsymbol{x}, t) v_i(\boldsymbol{x}, t)\, dv$$

To study its conservation, we need the material derivative expressed as follows:

$$\begin{aligned}
\frac{D\overline{m}_i(\omega, t)}{Dt} &= \frac{d}{dt}\int_\omega \rho v_i\, dv = \int_\omega \left(\frac{D(\rho v_i)}{Dt} + \rho v_i \frac{\partial v_m}{\partial x_m}\right) dv \\
&= \int_\omega \left(\frac{D\rho}{Dt} v_i + \rho \frac{D v_i}{Dt} + \rho v_i \frac{\partial v_m}{\partial x_m}\right) dv \\
&= \int_\omega \left(\rho \frac{D v_i}{Dt} + v_i \boxed{\left(\frac{D\rho}{Dt} + \rho \frac{\partial v_m}{\partial x_m}\right)}\right) dv \\
&= \int_\omega \rho a_i\, dv\,.
\end{aligned}$$

Local form of the mass conservation equal to 0

in vector form

$$\boxed{\begin{aligned}
\frac{D\overline{\boldsymbol{m}}(\omega, t)}{Dt} &= \int_\omega \rho(\boldsymbol{x}, t) \frac{D\boldsymbol{v}(\boldsymbol{x}, t)}{Dt}\, dv \\
&= \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{a}(\boldsymbol{x}, t)\, dv
\end{aligned}}$$

Continuum mechanics review: Dynamics

CONSERVATION OF ANGULAR MOMENTUM

In physics, the angular momentum of a particle with mass m is defined by:

$$\widehat{\boldsymbol{m}} = m\boldsymbol{x} \times \boldsymbol{v} \qquad \widehat{m}_i = m\varepsilon_{ijk}x_jv_k$$

For a part Π of a body $\mathcal{B}$ initially at $\mathcal{R}$ and currently at $\mathcal{R}_t$, the momentum is:

$$\widehat{\boldsymbol{m}}(\omega, t) = \int_\omega \rho(\boldsymbol{x}, t)\, \boldsymbol{x} \times \boldsymbol{v}(\boldsymbol{x}, t)\, dv$$

$$\widehat{m}_i(\omega, t) = \int_\omega \rho(\boldsymbol{x}, t)\varepsilon_{ijk}x_jv_k(\boldsymbol{x}, t)\, dv$$

To study its conservation, we need the material derivative expressed as follows:

$$\begin{aligned}
\frac{D\widehat{m}_i(\omega, t)}{Dt} &= \frac{d}{dt}\int_\omega \rho\varepsilon_{ijk}x_jv_k\, dv \\
&= \int_\omega \left(\frac{D(\rho\varepsilon_{ijk}x_jv_k)}{Dt} + \rho\varepsilon_{ijk}x_jv_k\frac{\partial v_m}{\partial x_m}\right) dv \\
&= \int_\omega \left(\frac{D\rho}{Dt}\varepsilon_{ijk}x_jv_k + \rho\varepsilon_{ijk}\left(\frac{Dx_j}{Dt}v_k + x_j\frac{Dv_k}{Dt} + x_jv_k\frac{\partial v_m}{\partial x_m}\right)\right) dv \\
&= \int_\omega \left(\rho\varepsilon_{ijk}x_j\frac{Dv_k}{Dt} + \boxed{\rho\varepsilon_{ijk}v_jv_k} + \varepsilon_{ijk}x_jv_k\left(\boxed{\frac{D\rho}{Dt} + \rho\frac{\partial v_m}{\partial x_m}}\right)\right) dv \\
&= \int_\omega \rho\varepsilon_{ijk}x_ja_k\, dv\,,
\end{aligned}$$

Product of symmetric and antisymmetric tensors equals 0

Local form of the mass Conservation is to equal 0

in vector form

$$\begin{aligned}
\frac{D\widehat{\boldsymbol{m}}(\omega, t)}{Dt} &= \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{x} \times \frac{D\boldsymbol{v}(\boldsymbol{x}, t)}{Dt}\, dv \\
&= \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{x} \times \boldsymbol{a}(\boldsymbol{x}, t)\, dv\,,
\end{aligned}$$

Continuum mechanics review: Dynamics

EULER’S LAWS OF MOTION

PRINCIPLE OF CONSERVATION OF MOMENTUM

The rate of change of the momentum of an arbitrary part Π *of a body* $\mathcal{B}$ *at time* t *is equal to the sum of the forces applied to* Π *at that instant.*

$$\frac{d}{dt}\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\,dv=\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)\,dv+\int_{\partial\omega}\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds\,.$$

With the conservation of momentum:

$$\boxed{\frac{d}{dt}\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\,dv=\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)\,dv}$$

$$\boxed{\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)\,dv=\int_{\omega}\rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)\,dv+\int_{\partial\omega}\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds}$$

It will lead to the equation of motion.

Continuum mechanics review: Dynamics

EULER'S LAWS OF MOTION

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

The rate of change of the angular momentum of an arbitrary part $\prod$ *of a body* $\mathcal{B}$ *at time* t *is equal to the moment (with respect to the origin) of the forces applied to* $\prod$ *at that instant.*

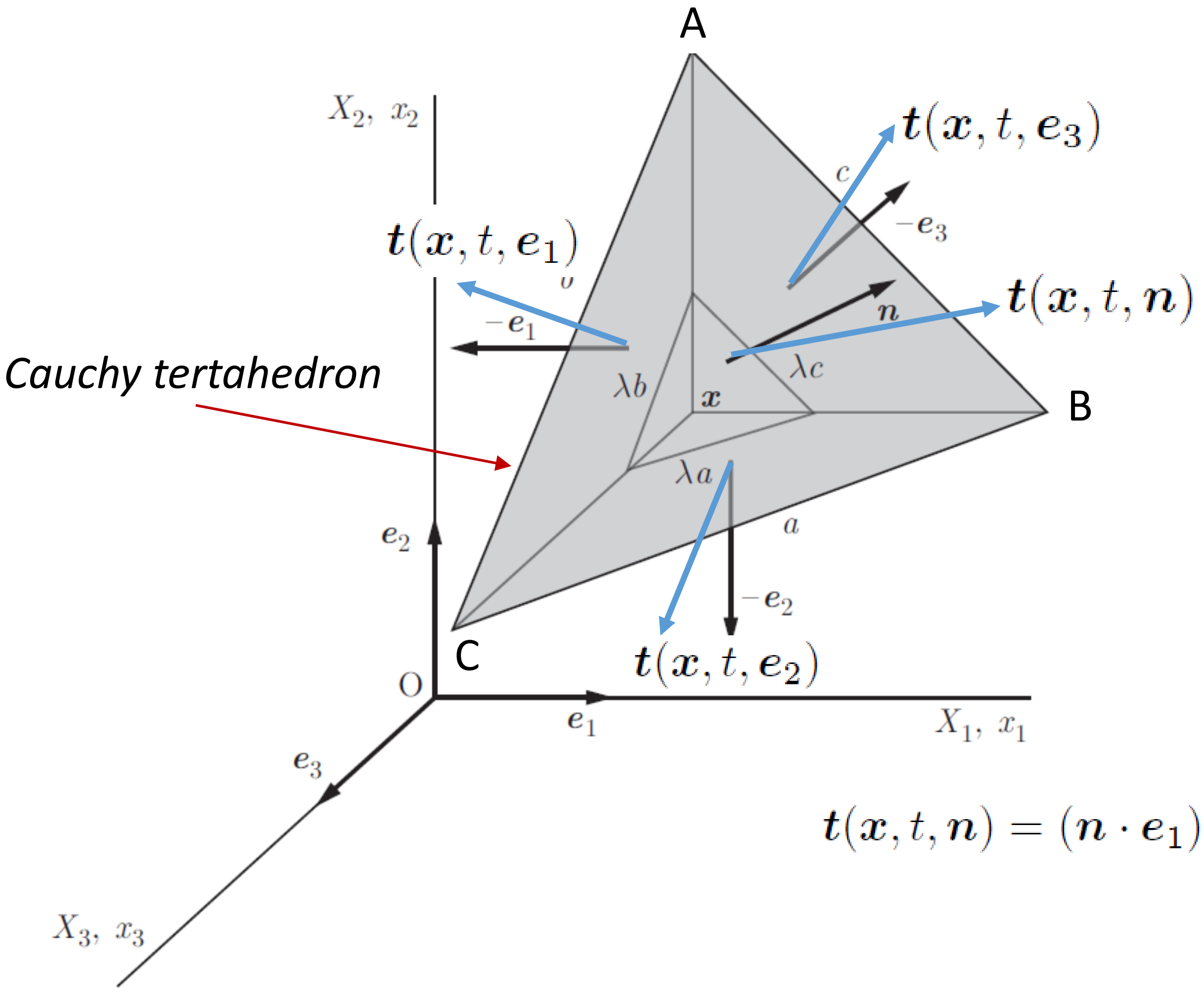
$$\frac{d}{dt}\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{v}(\boldsymbol{x},t)\,dv$$
$$=\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{b}(\boldsymbol{x},t)\,dv+\int_{\partial\omega} \boldsymbol{x}\times\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds$$

With the conservation of angular momentum: $\boxed{\frac{d}{dt}\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{v}(\boldsymbol{x},t)\,dv=\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{a}(\boldsymbol{x},t)\,dv}$

$$\boxed{\begin{aligned}&\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{a}(\boldsymbol{x},t)\,dv\\ &\quad=\int_{\omega} \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{b}(\boldsymbol{x},t)\,dv+\int_{\partial\omega} \boldsymbol{x}\times\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds\end{aligned}}$$

It will lead to the symmetry of the stress tensor.

Continuum mechanics review: Dynamics



Cauchy's Theorem

If the stress vector $\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$ is continuous with respect to $\boldsymbol{x}$ and if $\rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)$ and $\rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)$ are bounded, the principle of conservation of momentum implies that there exists a 2nd order stress tensor $\boldsymbol{\sigma}(\boldsymbol{x},t)$:

$$t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j$$

We apply the principle of conservation of momentum to the tertahedron shown on the left to obtain the stess vector on the ABC side in terms of the stress vectors on the three sides of the tertahedron as follows:

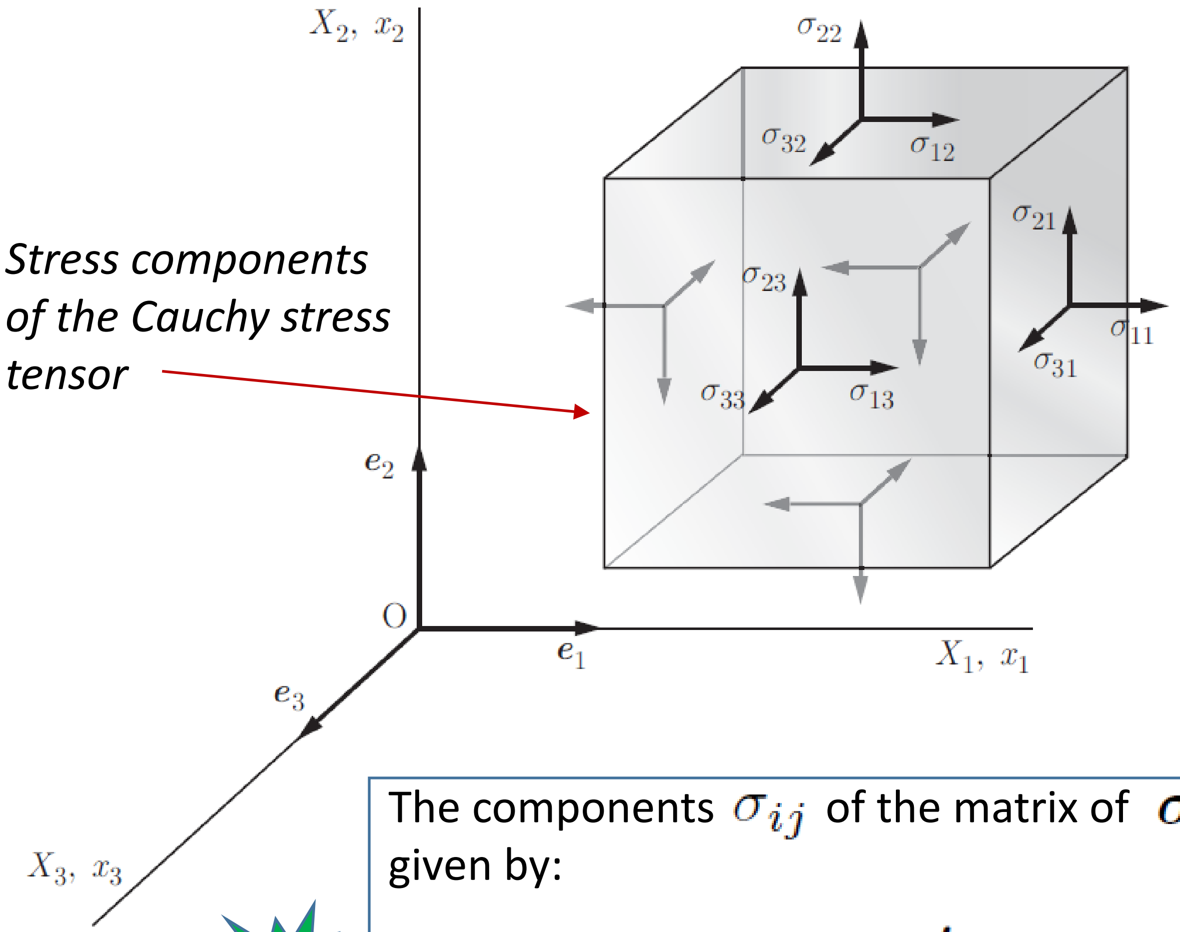
$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = (\boldsymbol{n}\cdot\boldsymbol{e}_1)\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_1) + (\boldsymbol{n}\cdot\boldsymbol{e}_2)\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_2) + (\boldsymbol{n}\cdot\boldsymbol{e}_3)\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_3)$$

$$\boxed{(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{v} = (\boldsymbol{b}\cdot\boldsymbol{v})\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{v})}$$

$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = \left(\underline{\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_1)\otimes\boldsymbol{e}_1 + \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_2)\otimes\boldsymbol{e}_2 + \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_3)\otimes\boldsymbol{e}_3}\right)\boldsymbol{n}$$

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \underline{\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_1)\otimes\boldsymbol{e}_1 + \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_2)\otimes\boldsymbol{e}_2 + \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{e}_3)\otimes\boldsymbol{e}_3}$$

Continuum mechanics review: Dynamics



Stress components of the Cauchy stress tensor

Consequences of the Cauchy's Theorem

The theorem expresses the linear dependence of $t_i(\boldsymbol{x}, t, \boldsymbol{n})$ with respect to the unit normal:

$$t_i(\boldsymbol{x}, t, \boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x}, t) n_j$$

1: when the stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is known, the stress vector acting at $\boldsymbol{x}$ on any surface with outgoing unit normal $\boldsymbol{n}$ is completely determined.
2: the state of stress at $\boldsymbol{x}$ (at time t) is characterized by the stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$.

The components σ_{ij} of the matrix of $\boldsymbol{\sigma}$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are given by:

$$\sigma_{ij} = \underset{\substack{\downarrow \\ \text{direction}}}{\boldsymbol{e}_i} \cdot \boldsymbol{\sigma} \underset{\substack{\downarrow \\ \text{surface}}}{\boldsymbol{e}_j} = \boldsymbol{e}_i \cdot \boldsymbol{t}_{\boldsymbol{e}_j} \longrightarrow$$

σ_{ij} is the component of the stress vector $\boldsymbol{t}_{\boldsymbol{e}_j}$ in the direction i acting on a spatial surface element whose unit normal is aligned in the direction of $\boldsymbol{e}_j$.

Continuum mechanics review: Dynamics

PRINCIPLE OF CONSERVATION OF MOMENTUM

Theorem

Suppose that the stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ is continuously differentiable with respect to $\boldsymbol{x}$, and that $\rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)$ and $\rho(\boldsymbol{x}, t)\boldsymbol{a}(\boldsymbol{x}, t)$ are continuous at $\boldsymbol{x}$. The principle of conservation of momentum:

$$\int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{a}(\boldsymbol{x}, t)\, dv = \int_\omega \rho(\boldsymbol{x}, t)\boldsymbol{b}(\boldsymbol{x}, t)\, dv + \boxed{\int_{\partial\omega} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds}$$

is satisfied if and only if, for an arbitrary point $\boldsymbol{x}$ of $\mathcal{R}_t$.

$$\boxed{div\sigma_{ij}(\boldsymbol{x},t) + \rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)} \quad \text{or} \quad \boxed{\sigma_{ij,j} + \rho b_i = \rho a_i}$$

$$\boxed{t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j}$$

$$\int_\omega \rho(\boldsymbol{x}, t)a_i(\boldsymbol{x}, t)\, dv = \int_\omega \rho(\boldsymbol{x}, t)b_i(\boldsymbol{x}, t)\, dv + \boxed{\int_{\partial\omega} \sigma_{ij}(\boldsymbol{x}, t)n_j\, ds}$$

$$\boxed{\int_\omega \left(\rho(\boldsymbol{x}, t)a_i(\boldsymbol{x}, t) - \rho(\boldsymbol{x}, t)b_i(\boldsymbol{x}, t) - \sigma_{ij,j}(\boldsymbol{x}, t)\right) dv = 0}$$

Continuum mechanics review: Dynamics

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

Theorem

Suppose that the stress tensor $\boldsymbol{\sigma}(\boldsymbol{x},t)$ is continuously differentiable with respect to $\boldsymbol{x}$, and that $\rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)$ and $\rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)$ are continuous at $\boldsymbol{x}$. The principle of conservation of angular momentum:

$$\int_\omega \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{a}(\boldsymbol{x},t)\,dv$$

$$= \int_\omega \rho(\boldsymbol{x},t)\boldsymbol{x}\times\boldsymbol{b}(\boldsymbol{x},t)\,dv + \boxed{\int_{\partial\omega} \boldsymbol{x}\times\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds}$$

implies the symmetry of the stress tensor, i. e. $\boxed{\boldsymbol{\sigma}^T = \boldsymbol{\sigma} \quad ; \quad \sigma_{ij} = \sigma_{ji}}$

$$\boxed{t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j}$$

$$\int_\omega \rho(\boldsymbol{x},t)\varepsilon_{ijk}x_j a_k(\boldsymbol{x},t)\,dv$$

$$= \int_\omega \rho(\boldsymbol{x},t)\varepsilon_{ijk}x_j b_k(\boldsymbol{x},t)\,dv + \boxed{\int_{\partial\omega} \varepsilon_{ijk}x_j\sigma_{km}(\boldsymbol{x},t)n_m\,ds}$$

Continuum mechanics review: Dynamics

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

Apply the divergence theorem to the last term of equation:

$$\int_{\omega} \rho(\boldsymbol{x},t)\varepsilon_{ijk}x_j a_k(\boldsymbol{x},t)\,dv$$

$$= \int_{\omega} \rho(\boldsymbol{x},t)\varepsilon_{ijk}x_j b_k(\boldsymbol{x},t)\,dv + \boxed{\int_{\partial\omega} \varepsilon_{ijk}x_j\sigma_{km}(\boldsymbol{x},t)n_m\,ds}$$

$$\boxed{\int_{\partial\omega} \varepsilon_{ijk}x_j\sigma_{km}(\boldsymbol{x},t)n_m\,ds}$$

$$= \int_{\omega} \varepsilon_{ijk}\big(x_{j,m}\sigma_{km}(\boldsymbol{x},t) + x_j\sigma_{km,m}(\boldsymbol{x},t)\big)\,dv$$

$$= \int_{\omega} \varepsilon_{ijk}\big(\sigma_{kj}(\boldsymbol{x},t) + x_j\sigma_{km,m}(\boldsymbol{x},t)\big)\,dv\,.$$

$$\int_{\omega} \varepsilon_{ijk}x_j\overbrace{\big(\rho(\boldsymbol{x},t)a_k(\boldsymbol{x},t) - \rho(\boldsymbol{x},t)b_k(\boldsymbol{x},t) - \sigma_{km,m}(\boldsymbol{x},t)\big)}^{\text{zero (equations of motion)}}\,dv$$

$$= \int_{\omega} \varepsilon_{ijk}\sigma_{kj}(\boldsymbol{x},t)\,dv\,.$$

$$\boxed{\int_{\omega} \varepsilon_{ijk}\sigma_{kj}(\boldsymbol{x},t)\,dv = 0}$$

$$\varepsilon_{ijk}\sigma_{kj} = \frac{1}{2}\varepsilon_{ijk}(\sigma_{kj}+\sigma_{jk}) - \frac{1}{2}\varepsilon_{ijk}(\sigma_{jk}-\sigma_{kj})$$

$$= -\frac{1}{2}\,\varepsilon_{ijk}(\sigma_{jk}-\sigma_{kj}) = 0$$

$$\boxed{\sigma_{jk} = \sigma_{kj}}$$

Continuum mechanics review: Dynamics

Properties of the Stress Tensor

The stress vector is given by the Cauchy's formula:

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \quad \text{or} \quad t_i = \sigma_{ij} n_j$$

When the stress vector acts along in the direction on the vector normal to the surface:

$$\boldsymbol{\sigma}\boldsymbol{n} = \lambda\boldsymbol{n} \quad \text{or} \quad \sigma_{ij} n_j = \lambda n_i$$

$$\det(\boldsymbol{\sigma} - \lambda\boldsymbol{I}) = 0 \quad \text{or} \quad \det(\sigma_{ij} - \lambda\delta_{ij}) = 0$$

$$\lambda^3 - \mathrm{I}_1(\boldsymbol{\sigma})\lambda^2 + \mathrm{I}_2(\boldsymbol{\sigma})\lambda - \mathrm{I}_3(\boldsymbol{\sigma}) = 0$$

This is the characteristic equation of an eigenvalue problem.

Here the stress invariants are given by:

$$\begin{aligned} \mathrm{I}_1(\boldsymbol{\sigma}) &= \mathrm{tr}\ \boldsymbol{\sigma} = \sigma_{ii} \\ \mathrm{I}_2(\boldsymbol{\sigma}) &= \frac{1}{2}\left((\mathrm{tr}\,\boldsymbol{\sigma})^2 - \mathrm{tr}\,\boldsymbol{\sigma}^2\right) = \frac{1}{2}\left((\sigma_{ii})^2 - \sigma_{mn}\sigma_{nm}\right) \\ \mathrm{I}_3(\boldsymbol{\sigma}) &= \det\boldsymbol{\sigma} = \varepsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} \end{aligned}$$

The three roots are real (stress tensor is symmetric) and are called principal stresses :
$\sigma_1,\ \sigma_2$ and σ_3
conventionally ordered as: $\sigma_1 \geq \sigma_2 \geq \sigma_3$

The corresponding principal eigenvectors are the associated principal directions (or planes) $\boldsymbol{n_i}$.

In terms of the principal stresses the invariants are:

$$\begin{aligned} \mathrm{I}_1(\boldsymbol{\sigma}) &= \sigma_1 + \sigma_2 + \sigma_3 \\ \mathrm{I}_2(\boldsymbol{\sigma}) &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ \mathrm{I}_3(\boldsymbol{\sigma}) &= \sigma_1\sigma_2\sigma_3\,. \end{aligned}$$

Continuum mechanics review: Dynamics

Properties of the Stress Tensor

The stress vector is given by the Cauchy's formula:

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \quad \text{or} \quad t_i = \sigma_{ij} n_j$$

The stress vector has a normal t_N and a tangential t_T component associated with shear stress:

$$t_N = \boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma}\boldsymbol{n} \quad \text{or} \quad t_N = n_i t_i = \sigma_{ij} n_i n_j$$

$$t_T = \left\| \boldsymbol{t} - (\boldsymbol{n} \cdot \boldsymbol{t})\boldsymbol{n} \right\| = \left\| (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{t} \right\|$$

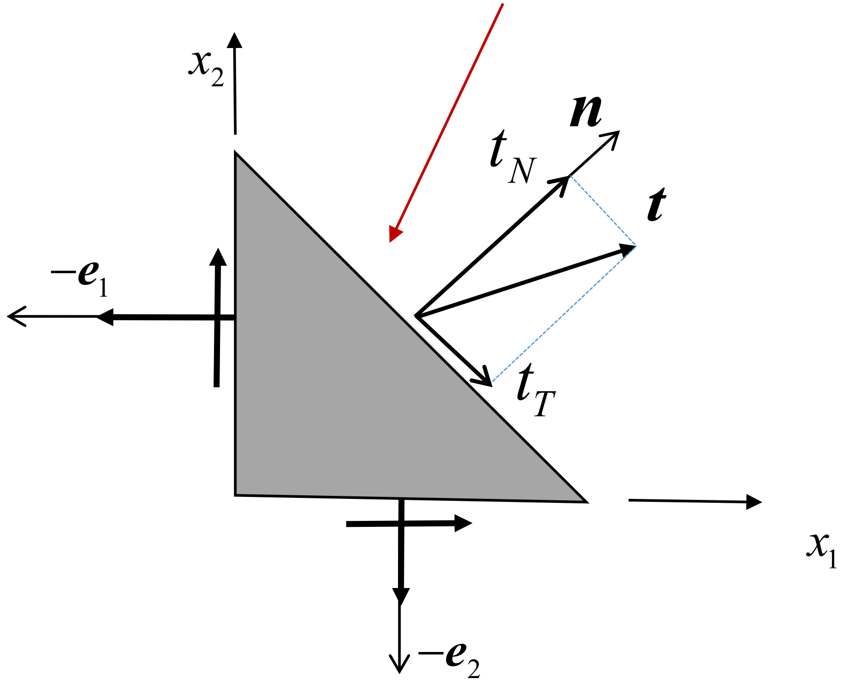
$$= (t_i t_i - t_N^2)^{1/2}$$

When the stress vector acts along the direction of the vector normal to the surface:

$$\boldsymbol{\sigma}\boldsymbol{n} = \lambda \boldsymbol{n} \quad \text{or} \quad \sigma_{ij} n_j = \lambda n_i$$

➡ $\lambda^3 - \mathrm{I}_1(\boldsymbol{\sigma})\lambda^2 + \mathrm{I}_2(\boldsymbol{\sigma})\lambda - \mathrm{I}_3(\boldsymbol{\sigma}) = 0$

Normal and tangential stresses on oblique plane, in two dimensional load



Continuum mechanics review: Dynamics

Stress Vector at a plane $\boldsymbol{n}$

The stress vector is given by the Cauchy's formula:

$$\boldsymbol{t} = \boldsymbol{\sigma n} \quad \text{or} \quad t_i = \sigma_{ij} n_j$$

The stress vector has a normal t_N and a tangential t_T component associated with shear stress:

$$t_N = \boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\sigma n} \text{ or } t_N = n_i t_i = \sigma_{ij} n_i n_j$$

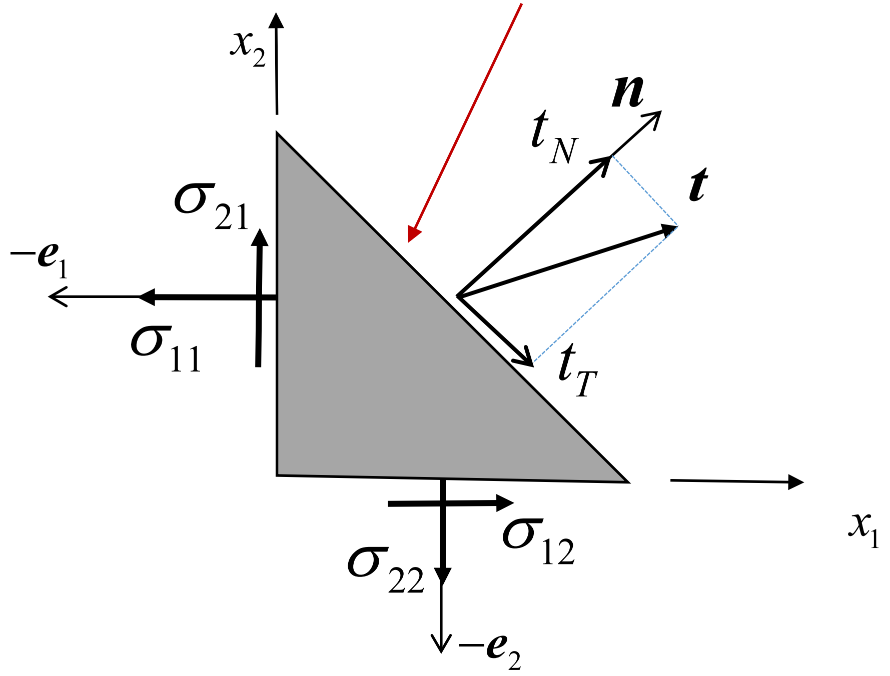
$$t_T = \left\| \boldsymbol{t} - (\boldsymbol{n} \cdot \boldsymbol{t})\boldsymbol{n} \right\| = \left\| (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{t} \right\|$$
$$= (t_i t_i - t_N^2)^{1/2}$$

The expression:

$$t_i = \sigma_{ij} n_j$$

is used to calculate the stress vector $\boldsymbol{t}$ at any given plane with normal $\boldsymbol{n}$ passing by a point $\boldsymbol{x}$

Stress vector $\boldsymbol{t}$ at $\boldsymbol{n}$, an oblique plane, in two dimensional load



Octahedral plane: It is a special plane: Its unit normal vector has equal components with respect to all three principal directions

$$\boldsymbol{m} = (1/\sqrt{3})\boldsymbol{n}_1 + (1/\sqrt{3})\boldsymbol{n}_2 + (1/\sqrt{3})\boldsymbol{n}_3$$

Continuum mechanics review: Dynamics

Transformation of the stress tensor

The components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$ form a 3x3 matrix given by (relative to $\{e_1, e_2, e_3\}$) :

$$\sigma_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{\sigma} \boldsymbol{e}_j$$

Consider another orthogonal basis $\{e_1', e_2', e_3'\}$ obtained by rotation of $\{e_1, e_2, e_3\}$:

$$\boldsymbol{e}_i' = c_{ij} \boldsymbol{e}_j \qquad (i = 1, 2, 3)$$

➡ $$\sigma_{ij}' = \boldsymbol{e}_i' \cdot \boldsymbol{\sigma} \boldsymbol{e}_j' \\ = c_{im} c_{jn} \boldsymbol{e}_m \cdot \boldsymbol{\sigma} \boldsymbol{e}_n = c_{im} c_{jn} \sigma_{mn}$$

In matrix form the transformation is: $\left[\sigma^{'}\right] = [C][\sigma][C]^T$

The elements of $[C]$ are the direction cosines of the $\{e_1', e_2', e_3'\}$ with respect to $\{e_1, e_2, e_3\}$ and

$$[C][C]^T = [C][C]^{-1} = [I]$$

Deviatoric stress tensor

It is often useful to decompose $\boldsymbol{\sigma}$ in two components as follows:

$$\boldsymbol{\sigma} = \boldsymbol{s} + \sigma_0 \boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = s_{ij} + \sigma_0 \delta_{ij}$$

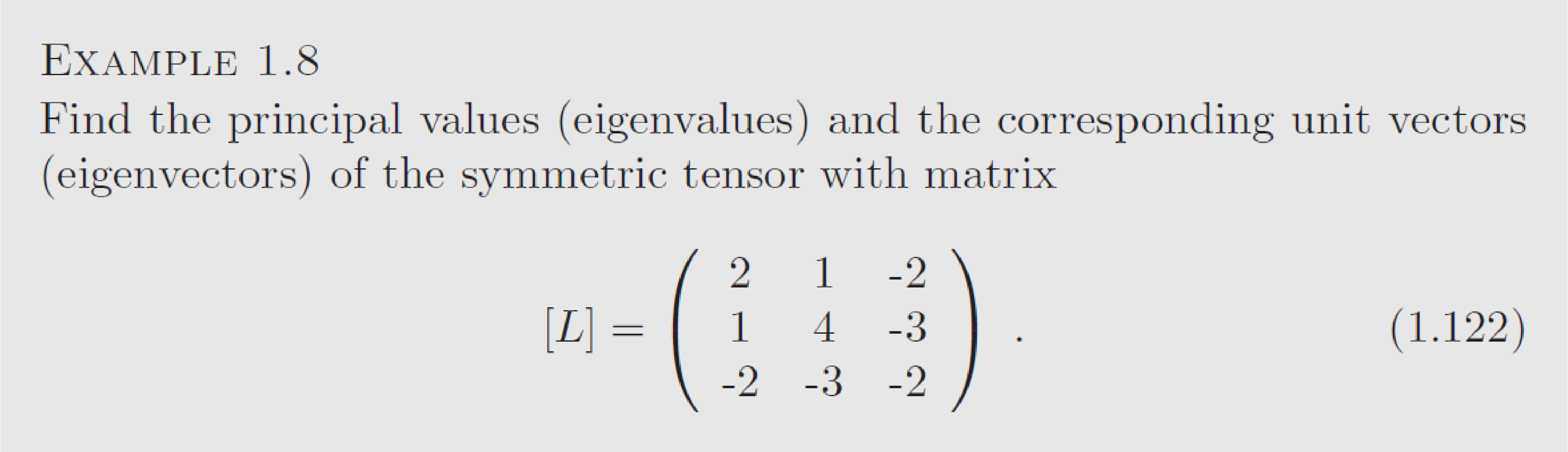
where $\boldsymbol{s} = \boldsymbol{\sigma} - \sigma_0 \boldsymbol{I}$

$$\sigma_0 = \frac{1}{3} I_1(\boldsymbol{\sigma}) = \frac{1}{3} \sigma_{kk}$$

Tensor $\boldsymbol{s}$ is called deviatoric stress tensor associated with $\boldsymbol{\sigma}$ and the property:

$$\operatorname{tr} \boldsymbol{s} = s_{ii} = 0$$

Stress Analysis: Principal stresses



Example 1.8

Find the principal values (eigenvalues) and the corresponding unit vectors (eigenvectors) of the symmetric tensor with matrix

$$[L] = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 4 & -3 \\ -2 & -3 & -2 \end{pmatrix} . \qquad (1.122)$$

Using the expressions below, we determine the invariants, characteristic equation and its roots:

$$I_1(\boldsymbol{L}) = L_{ii} = \operatorname{tr} \boldsymbol{L}$$

$$\begin{aligned} I_2(\boldsymbol{L}) &= \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} + \begin{vmatrix} L_{22} & L_{23} \\ L_{32} & L_{33} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} \\ L_{31} & L_{33} \end{vmatrix} \\ &= \frac{1}{2}\left(L_{ii}L_{jj} - L_{ij}L_{ji}\right) \\ &= \frac{1}{2}\left((\operatorname{tr} \boldsymbol{L})^2 - \operatorname{tr}(\boldsymbol{L}\boldsymbol{L})\right) = \frac{1}{2}\left((\operatorname{tr} \boldsymbol{L})^2 - \operatorname{tr}(\boldsymbol{L}^2)\right) \end{aligned}$$

$$I_3(\boldsymbol{L}) = \varepsilon_{ijk} L_{i1} L_{j2} L_{k3} = \det \boldsymbol{L} \, .$$

$$I_1(\boldsymbol{L}) = 4, \quad I_2(\boldsymbol{L}) = -18, \quad I_3(\boldsymbol{L}) = -36,$$

$$\lambda^3 - 4\lambda^2 - 18\lambda + 36 = 0.$$

$$\lambda_1 = 6, \quad \lambda_2 = 1.65, \quad \lambda_3 = -3.65.$$

For the principal directions see the rest of the example in the book (Botsis & Deville)

Stress Analysis: Stress vector on a plane

We consider the tensor of stresses,

$$[\sigma] = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 4 & -3 \\ -2 & -3 & -2 \end{pmatrix}$$

To calculate the stress vector at a point belonging to the plane with outward normal given by:

$$\boldsymbol{n} = 2\boldsymbol{e}_1/3 + 2\boldsymbol{e}_2/3 - \boldsymbol{e}_3/3:$$

we use the Cauchy expression

$$t_i = \sigma_{ij} n_j$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 4 & -3 \\ -2 & -3 & -2 \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 8 \\ 13 \\ -8 \end{pmatrix}$$

We can also calculate the normal and tangential components on the plane using the expressions

$$t_N = n_i t_i = 5.55$$

$$t_T = \left(t_i t_i - t_N^2\right)^{1/2} = 4.56$$

Stress Analysis: Octahedral plane

We take next the stress tensor with its principal values

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 1.65 & 0 \\ 0 & 0 & -3.65 \end{pmatrix}$$

and consider a coordinate system defined by the principal directions and the following plane with normal:

$$\boldsymbol{m} = (1/\sqrt{3})\boldsymbol{n}_1 + (1/\sqrt{3})\boldsymbol{n}_2 + (1/\sqrt{3})\boldsymbol{n}_3.$$

We calculate the stress vector on that plane using

$$t_i = \sigma_{ij} n_j$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1.65 & 0 \\ 0 & 0 & -3.65 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 6 \\ 1.65 \\ -3.65 \end{pmatrix}$$

The plane so defined is called

OCTAHEDRAL PLANE

$$t_T = \left(t_i t_i - t_N^2\right)^{1/2} = 0.94.$$

$$t_N = n_i t_i = 1.33$$

On that plane the normal and tangential components are

Stress Analysis: Octahedral stresses

On the octahedral plane we have two stress components (octahedral stresses):

$$\begin{aligned} t_N &= I_1(\boldsymbol{\sigma})/3 \\ t_T &= \frac{1}{3}\sqrt{2I_1^2(\boldsymbol{\sigma}) - 6I_2(\boldsymbol{\sigma})}\,. \end{aligned}$$

Octahedral normal

Octahedral shear

$$t_T = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sqrt{\frac{2}{3}I_2(\boldsymbol{s})}$$

Note that the latter expression is proportional to the von Mises stress:

$$\sigma_e = \left[\frac{1}{2}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)\right]^{1/2} = \sqrt{\frac{1}{3}I_2(\boldsymbol{s})}$$

Continuum mechanics review: Dynamics

Equilibrium equations for a continuous medium

We set the acceleration zero in the equations of motion:

$$div\sigma_{ij}(\boldsymbol{x},t)+\rho(\boldsymbol{x},t)\boldsymbol{b}(\boldsymbol{x},t)=\rho(\boldsymbol{x},t)\boldsymbol{a}(\boldsymbol{x},t)$$

to obtain the equilibrium eqs:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+\rho b_1=0$$
$$\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+\rho b_2=0$$
$$\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+\rho b_3=0$$

or $$\frac{\partial\sigma_{ij}}{\partial x_j}+\rho b_i=0$$

(ρ is the mass density and b_i the volume force density)

Uniform tension or compression

Tension is applied in direction 1. $\boldsymbol{\sigma}$ is given by:

$$\boldsymbol{\sigma}=\sigma\boldsymbol{n}_1\otimes\boldsymbol{n}_1 \quad \text{or} \quad [\sigma]=\begin{pmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

(σ is a constant)

This state of stress is encountered in a prismatic bar parallel to the direction 1.

More general uniform tensiom or compression in a direction $\boldsymbol{m}$ is expressed as:

$$\boldsymbol{\sigma}=\sigma(\boldsymbol{m}\otimes\boldsymbol{m}) \quad \text{or} \quad \sigma_{ij}=\sigma m_i m_j$$

(σ is a constant).

Continuum mechanics review: Dynamics

Uniform shear

A uniform shear is applied in direction 1 on the planes perpendicular to $\boldsymbol{e}_2$. The stress tensor becomes:

$$\boldsymbol{\sigma} = \tau(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) \quad \text{where } \tau \geq 0$$

or

$$[\sigma] = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation becomes:

$$\lambda(\lambda^2 - \tau^2) = 0$$

With solutions as the principal stresses:

$$\sigma_1 = \tau, \; \sigma_2 = 0, \sigma_3 = -\tau$$

And corresponding principal directions:

$$\boldsymbol{n}_1 = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}, \qquad \boldsymbol{n}_2 = \boldsymbol{e}_3$$

$$\boldsymbol{n}_3 = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$$

Uniform tension or compression

Tension is applied in direction 1. $\boldsymbol{\sigma}$ is given by:

$$\boldsymbol{\sigma} = \sigma \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 \qquad \text{or} \qquad [\sigma] = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(σ is a constant)

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More general uniform tension or compression in a direction $\boldsymbol{m}$ is expressed as:

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(σ is a constant)

Continuum mechanics review: Dynamics

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or

$$[\sigma] = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The characteristic equation becomes:

$$\lambda(\lambda^2 - \tau^2) = 0$$

With solutions as the principal stresses:

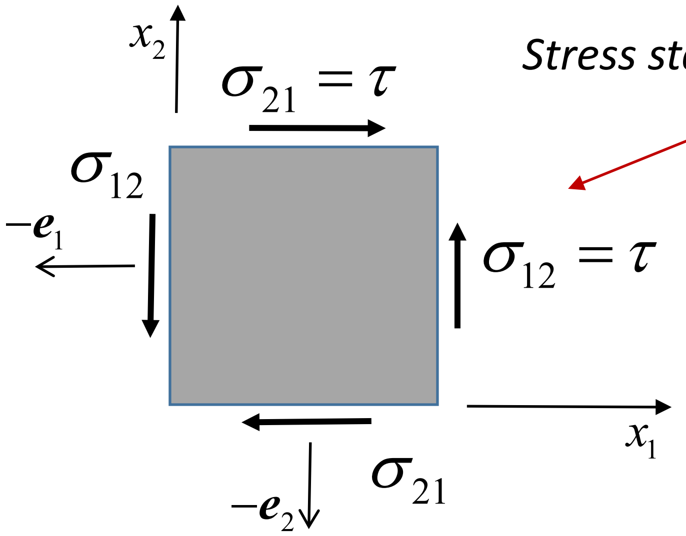
$$\sigma_1 = \tau, \; \sigma_2 = 0, \sigma_3 = -\tau$$

And corresponding principal directions:

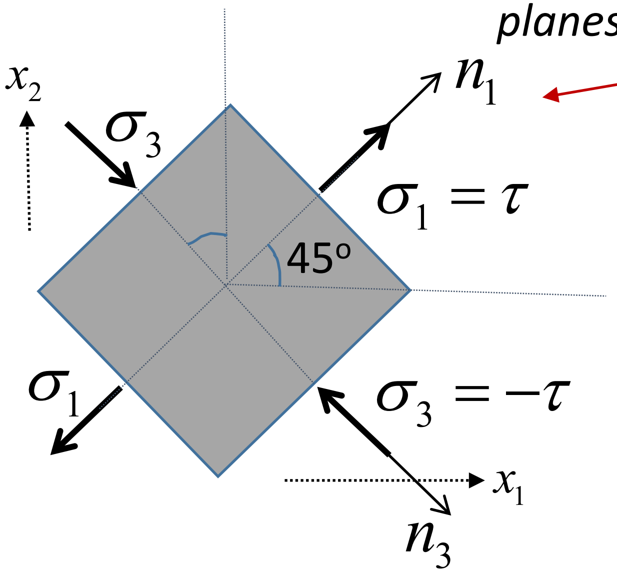
$$\boldsymbol{n}_1 = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}, \qquad \boldsymbol{n}_2 = \boldsymbol{e}_3$$

$$\boldsymbol{n}_3 = (\boldsymbol{e}_1 - \boldsymbol{e}_2)/\sqrt{2}$$

Stress state in uniform shear



Principal stresses and their planes in uniform shear



Continuum mechanics review: Dynamics

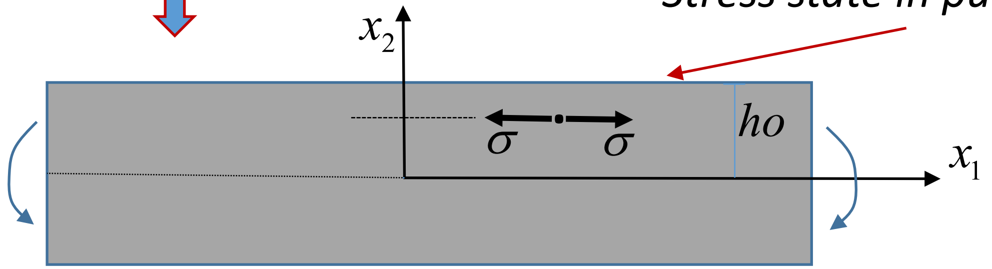
Pure Bending

Assuming zero body forces, the stress at x_2 is (α and h_0 are constants):

$$\boldsymbol{\sigma} = \alpha(x_2 - h_0)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$$

or

$$[\sigma] = \begin{pmatrix} \alpha(x_2 - h_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Stress state in pure bending

Hydrostatic pressure

Here the stress tensor takes the form:

$$\boldsymbol{\sigma} = -p(\boldsymbol{x})\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = -p(\boldsymbol{x})\,\delta_{ij}$$

With the equilibrium equations reducing to:

$$-\nabla p + \rho \boldsymbol{b} = 0 \quad \text{or} \quad -p_{,i} + \rho b_i = 0$$

With the body force zero, a body subjected to a constant hydrostatic pressure is in equilibrium since:

$$-\nabla p = 0$$

Continuum mechanics review: Dynamics

Plane stress:

It defined when $\sigma_{11},\ \sigma_{22},\ \sigma_{12} = \sigma_{21}$ may not be zero while the other components are zero. We have:

$$\boldsymbol{\sigma} = \sigma_{11}\boldsymbol{e_1} \otimes \boldsymbol{e_1} + \sigma_{22}\boldsymbol{e_2} \otimes \boldsymbol{e_2} + \sigma_{12}(\boldsymbol{e_1} \otimes \boldsymbol{e_2} + \boldsymbol{e_2} \otimes \boldsymbol{e_1})$$

Or

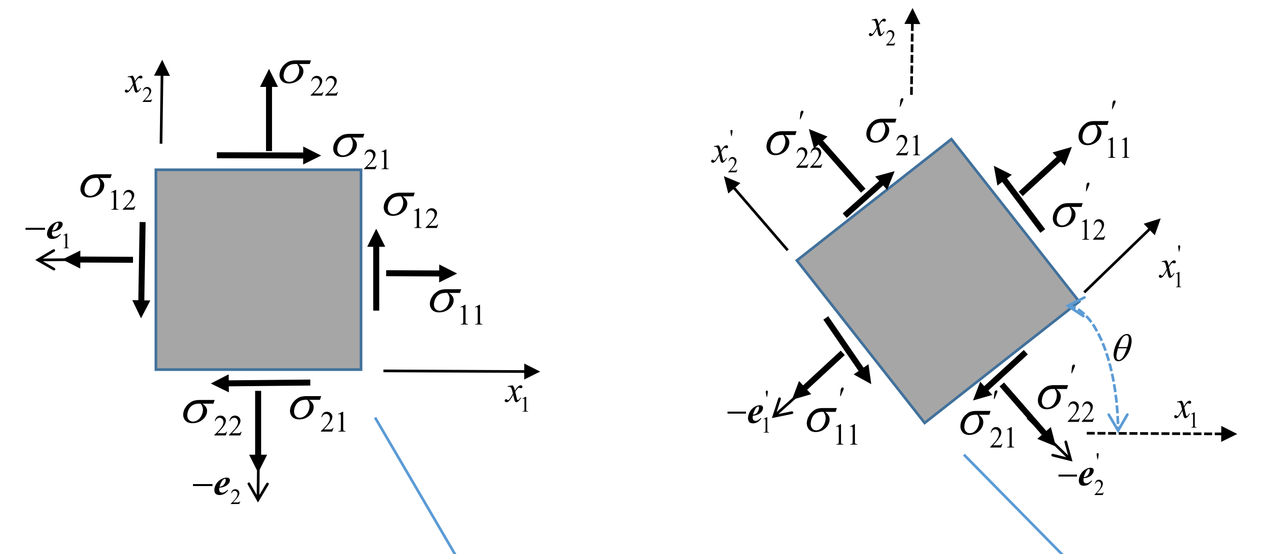
$$[\sigma] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The stresses are only functions of x_1 and x_2. Assuming zero body forces, the equilibrium equations become:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Stress components at a point in plane stress in two coordinate systems



$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \sigma_{11}^{'} & \sigma_{12}^{'} \\ \sigma_{21}^{'} & \sigma_{22}^{'} \end{pmatrix}$$

$(\sigma_{12} = \sigma_{21})$

$$\sigma_{11}^{'} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta$$

$$\sigma_{22}^{'} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\cos\theta\sin\theta$$

$$\sigma_{12}^{'} = (\sigma_{22} - \sigma_{11})\cos\theta\sin\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)$$

Note that a Mohr's circle for stress analysis is the geometrical solution/intrepretation of these equations.

From $\left[\sigma^{'}\right] = [C][\sigma][C]^T$

Continuum mechanics review: Dynamics

PIOLA-KIRCHHOFF STRESS TENSORS

The Cauchy stress tensor $\boldsymbol{\sigma}$, is expressed with respect to the current configuration $\mathcal{R}$ i.,e. it is the real stress.

The principles of momentum and angular momentum are formulated with respect to the current configurtion.

Problems in solid mechanics require a formulation with respect to the initial configuration $\mathcal{R}_0$.

This is because (a) it is difficult to know the deformed condition of a solid beforehand, (b) it is more convenient to analyze the experimental response of a solid with respect to its undeformed configuration.

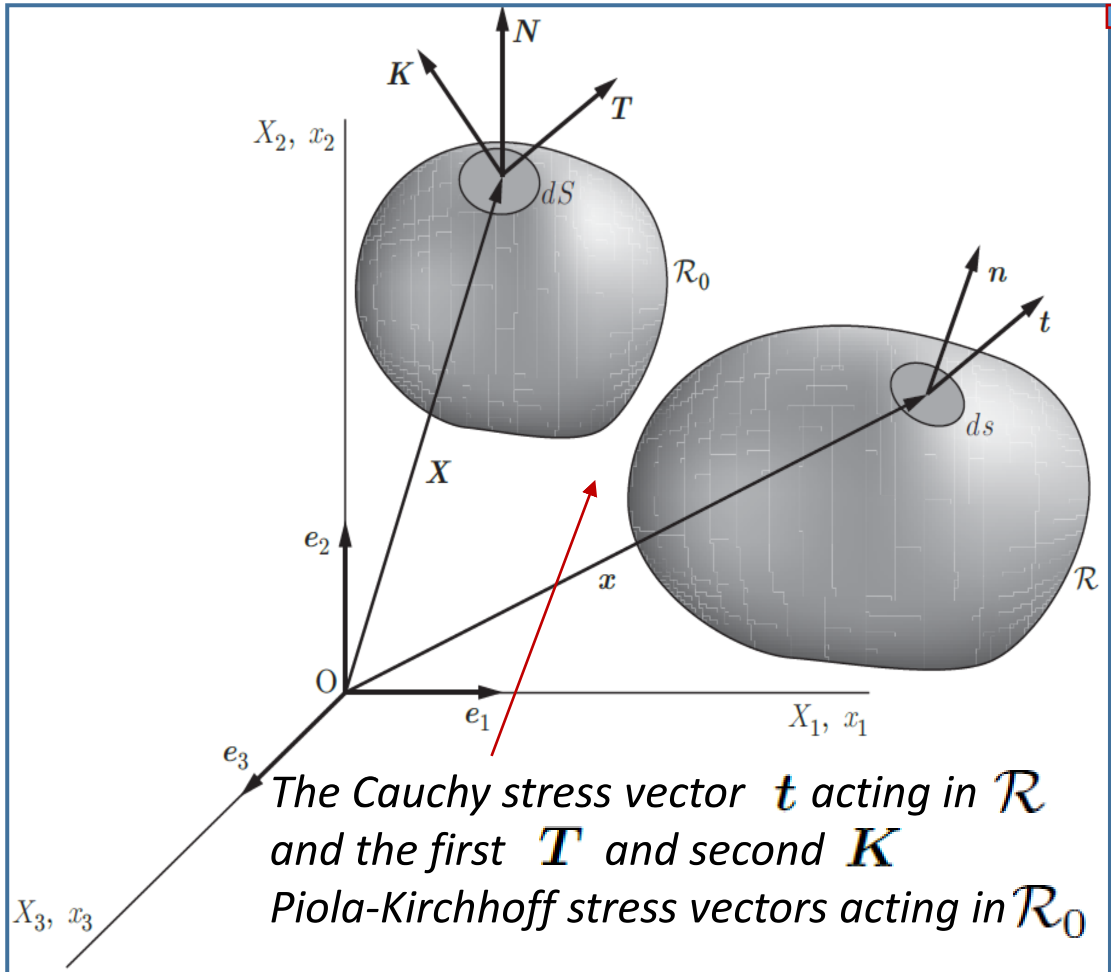
There is not simply a change of variables in the equations of motion and the Cauchy stress components using:

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t)$$

Measurements of stresses in the undeformed configuration have been proposed for the study of problems in solid mechanics.

These are the first and second Piola-Kirchhoff stress tensors.

Continuum mechanics review: Dynamics



The Cauchy stress vector $\boldsymbol{t}$ *acting in* $\mathcal{R}$ *and the first* $\boldsymbol{T}$ *and second* $\boldsymbol{K}$ *Piola-Kirchhoff stress vectors acting in* $\mathcal{R}_0$

Here $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ is the Cauchy stress vector acting on the actual surface element $\boldsymbol{n}\, ds$ at $\boldsymbol{x}$.

To this vector we associate the vector $\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})$ Called the first Piola-Kirchhoff stress vector, to the corresponding reference surface element $\boldsymbol{N}\, dS$, and related to $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ as follows:

$$\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N}(\boldsymbol{X}))\, dS = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}(\boldsymbol{x}, t))\, ds$$

dS and ds are positive. Thus, $\boldsymbol{T}$ and $\boldsymbol{t}$ have the same direction but $\|\boldsymbol{T}\|$ and $\|\boldsymbol{t}\|$ are not generally the same.

Note that the stress vector is not real (often called pseudo-stress).

Using Cauchy's relation $t_i(\boldsymbol{x}, t, \boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x}, t) n_j$

and Nanson's formula $\boldsymbol{n}\, ds = J\boldsymbol{F}^{-T}\boldsymbol{N}\, dS$

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})\, dS &= \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds = \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n}\, ds \\ &= J(\boldsymbol{X}, t)\boldsymbol{\sigma}(\chi(\boldsymbol{X}, t), t)\boldsymbol{F}^{-T}\boldsymbol{N}\, dS\end{aligned}$$

Continuum mechanics review: Dynamics

PIOLA-KIRCHHOFF STRESS TENSORS

$$\boxed{\begin{aligned} \boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N})\,dS &= \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds = \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}\,ds \\ &= J(\boldsymbol{X},t)\boldsymbol{\sigma}\big(\chi(\boldsymbol{X},t),t\big)\boldsymbol{F}^{-T}\boldsymbol{N}\,dS \end{aligned}} \quad \Rightarrow \quad \boldsymbol{T}(\boldsymbol{X},t,\boldsymbol{N}) = \boldsymbol{P}(\boldsymbol{X},t)\boldsymbol{N}$$

The First Piola-Kirchhoff stress tensor defined as:

$$\boldsymbol{P}(\boldsymbol{X},t) = J(\boldsymbol{X},t)\boldsymbol{\sigma}\big(\chi(\boldsymbol{X},t),t\big)\boldsymbol{F}^{-T}$$

Using the symmetry of the Cauchy's stress tensor $\sigma = \sigma^T$ it is shown below that: $\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$

$$\boxed{\begin{aligned} &\boldsymbol{P} = J\sigma\boldsymbol{F}^{-T} \quad \Rightarrow \quad \boldsymbol{P}\boldsymbol{F}^T = J\sigma \\ &\left(\boldsymbol{P}\boldsymbol{F}^T\right)^T = \boldsymbol{F}\boldsymbol{P}^T = J\left(\sigma\right)^T = J\sigma \\ &\Rightarrow \boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T \end{aligned}}$$

Tensor $\boldsymbol{P}$ is not symmetric and the principle of conservation of angular momentum is satisfied when this relation is met.

Continuum mechanics review: Dynamics

Objectivity of the Tensor $\boldsymbol{\sigma}$

We consider Cauchy's relation $t_i(\boldsymbol{x},t,\boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x},t)n_j$ as seen by two observers R and R^*.

We assume that vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are objective and are transformed as:

$$\boldsymbol{t}^* = \boldsymbol{Q}\boldsymbol{t} \quad ; \quad \boldsymbol{n}^* = \boldsymbol{Q}\boldsymbol{n}$$

From $\boldsymbol{t}^* = \boldsymbol{\sigma}^*\boldsymbol{n}^*$ we have $\boldsymbol{Q}\boldsymbol{t} = \boldsymbol{\sigma}^*\boldsymbol{Q}\boldsymbol{n}$

We multiply $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$ by $\boldsymbol{Q}$ to obtain: $\boldsymbol{Q}\boldsymbol{t} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{n}$

Comparing the two last results we have:

$$\boldsymbol{\sigma}^* = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$$

→ the Cauchy's stress tensor is objective.

Objectivity of the Tensor $\boldsymbol{P}$

To check the objectivity of $\boldsymbol{P}$ we start with

$$\boldsymbol{P}^*\boldsymbol{F}^{*T} = J^*\boldsymbol{\sigma}^*$$

and use

$$\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F} \quad ; \quad J^* = J \quad ; \quad \boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

$$\boldsymbol{P}^*(\boldsymbol{Q}\boldsymbol{F})^T = J\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$$

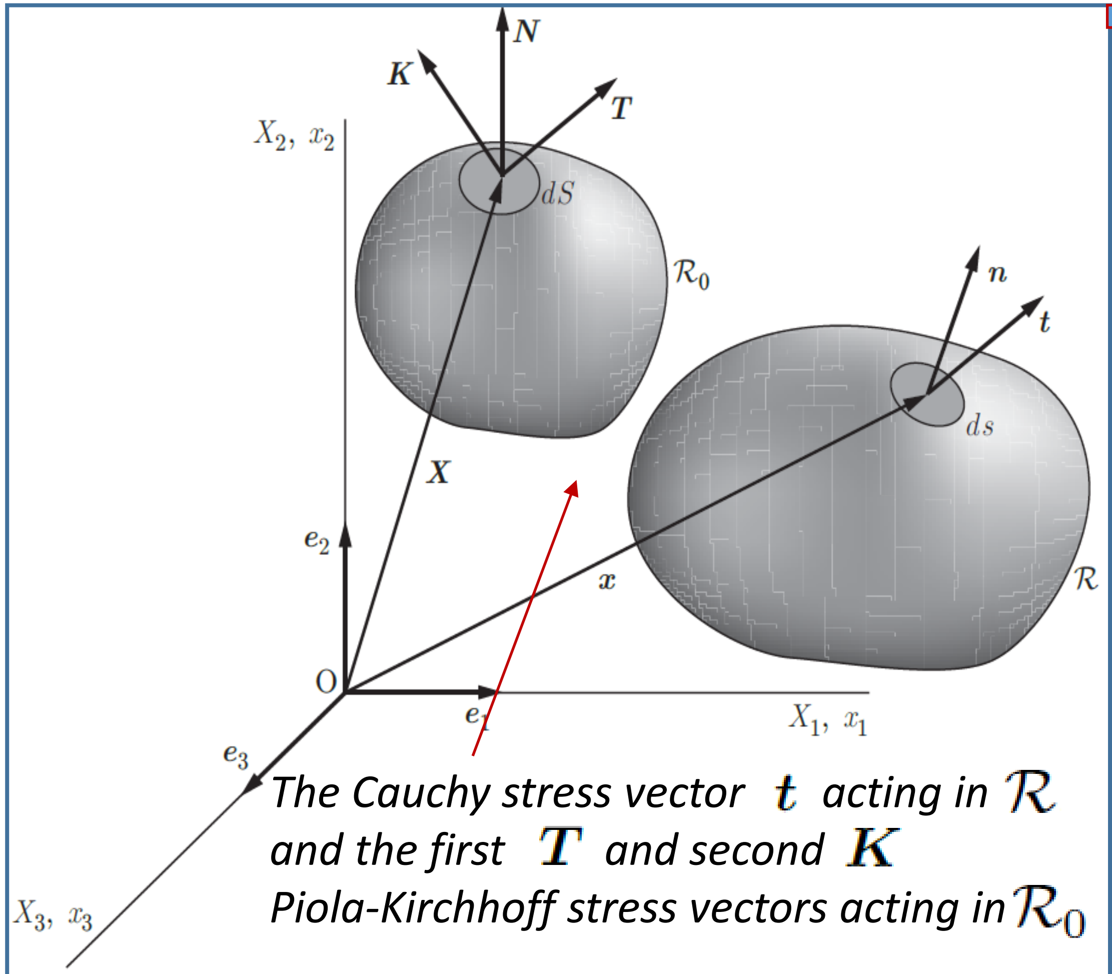
$$\boldsymbol{P}^*\boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{Q}J\boldsymbol{\sigma}\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{P}\boldsymbol{F}^T\boldsymbol{Q}^T$$

$$\boldsymbol{P}^* = \boldsymbol{Q}\boldsymbol{P}\,.$$

↓

The first Piola-Kirchhoff stress tensor is not objective.

Continuum mechanics review: Dynamics



The Cauchy stress vector $\boldsymbol{t}$ *acting in* $\mathcal{R}$ *and the first* $\boldsymbol{T}$ *and second* $\boldsymbol{K}$ *Piola-Kirchhoff stress vectors acting in* $\mathcal{R}_0$

Here $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ is the Cauchy stress vector acting on the actual surface element $\boldsymbol{n}\,ds$ at $\boldsymbol{x}$.
To this vector we associate the vector called the second Piola-Kirchhoff stress vector, to the corresponding reference surface element $\boldsymbol{N}\,dS$, and related to $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ as follows:

$$\boldsymbol{K}(\boldsymbol{X}, t, \boldsymbol{N})\,dS = \boldsymbol{F}^{-1}(\boldsymbol{X}, t)\,\boldsymbol{t}\big(\chi(\boldsymbol{X}, t),\ t,\ \boldsymbol{n}(\boldsymbol{X}, t)\big)\,ds$$

Here $\boldsymbol{K}$ expresses the conact force per unit reference surface transformed by $\boldsymbol{F}^{-1}$: Expressing

$$\boldsymbol{K}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{S}(\boldsymbol{X}, t)\boldsymbol{N}$$

We can define the second Piola-Kirchhoff tensor $\boldsymbol{S}$ given below:

$$\boldsymbol{S}(\boldsymbol{X}, t) = J(\boldsymbol{X}, t)\boldsymbol{F}^{-1}(\boldsymbol{X}, t)\boldsymbol{\sigma}\big(\chi(\boldsymbol{X}, t), t\big)\boldsymbol{F}^{-T}(\boldsymbol{X}, t)$$
$$= \boldsymbol{F}^{-1}(\boldsymbol{X}, t)\boldsymbol{P}(\boldsymbol{X}, t)\,.$$ which is symmetric

Using Cauchy's relation: $t_i(\boldsymbol{x}, t, \boldsymbol{n}) = \sigma_{ij}(\boldsymbol{x}, t)n_j$

and Nanson's formula: $\boldsymbol{n}\,ds = J\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$

Continuum mechanics review: Dynamics

Linearization of the Stress Tensors

It is important to check what are the effects of the Kinematic linearization on the three stress tensors.

From the relation:

$$\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$$

We express the tensor $\boldsymbol{P}$ in index form as:

$$P_{mk} = F_{mi}(P_{ij})^T(F_{jk})^{-T} = F_{mi}P_{ji}F_{kj}^{-1}$$

Using: $F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j}$; $F_{ij}^{-1} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$

and $\frac{\partial U_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + O(\varepsilon^2) \simeq \frac{\partial u_i}{\partial x_j}$

$$P_{mk} = P_{km} - P_{jm}\frac{\partial U_k}{\partial X_j} + P_{ki}\frac{\partial U_m}{\partial X_i} - P_{ji}\frac{\partial U_m}{\partial X_i}\frac{\partial U_k}{\partial X_j}$$

Similarly the second Piolla-Kirchhoff stress tensor:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}$$

takes the form in index notation:

$$S_{ij} = F_{ik}^{-1}P_{kj} = \left(\delta_{ik} - \frac{\partial U_i}{\partial X_k}\right)P_{kj} = P_{ij} - P_{kj}\frac{\partial U_i}{\partial X_k}$$

For the Cauchy stress tensor we express it using:

$$\boldsymbol{P} = J\sigma\boldsymbol{F}^{-T}$$

as $\sigma_{ij} = J^{-1}P_{ik}(F_{kj})^T = J^{-1}P_{ik}F_{jk}$

Continuum mechanics review: Dynamics

Linearization of the Stress Tensors

$$\sigma_{ij} = J^{-1}P_{ik}(F_{kj})^T = J^{-1}P_{ik}F_{jk}$$

Using $$F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j} \; ; \; F^{-1}_{ij} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

$$F_{ij} = \delta_{ij} + O(\varepsilon) \; ; \; F^{-1}_{ij} = \delta_{ij} - O(\varepsilon)$$

$$J \approx 1 + O(\varepsilon)$$

$$\Downarrow$$

$$\sigma_{ij} = J^{-1}P_{ik}\left(\delta_{jk} + \frac{\partial U_j}{\partial X_k}\right)$$

$$= J^{-1}(P_{ij} + P_{ik}\frac{\partial U_j}{\partial X_k}) \approx P_{ij} + P_{ik}\frac{\partial U_j}{\partial X_k}$$

If we neglect the terms with the displacement gradient on the expressions:

$$P_{mk} = P_{km} - P_{jm}\frac{\partial U_k}{\partial X_j} + P_{ki}\frac{\partial U_m}{\partial X_i} - P_{ji}\frac{\partial U_m}{\partial X_i}\frac{\partial U_k}{\partial X_j}$$

$$S_{ij} = F^{-1}_{ik}P_{kj} = \left(\delta_{ik} - \frac{\partial U_i}{\partial X_k}\right)P_{kj} = P_{ij} - P_{kj}\frac{\partial U_i}{\partial X_k}$$

$$\sigma_{ij} = J^{-1}P_{ik}\left(\delta_{jk} + \frac{\partial U_j}{\partial X_k}\right)$$

$$= J^{-1}(P_{ij} + P_{ik}\frac{\partial U_j}{\partial X_k}) \approx P_{ij} + P_{ik}\frac{\partial U_j}{\partial X_k}$$

$$\Downarrow$$

$$P_{mk} \approx P_{km} \qquad S_{ij} \approx P_{ij} \qquad \sigma_{ij} \approx P_{ij}$$